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Spinor Wave Equation of Electromagnetic Field

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Abstract In this paper, we give two spinor wave equations of free electromagnetic field, corresponding to the reducibility and irreducibility representations $D^{10}+D^{01}$ and D^{10} of the proper Lorentz group, which are the differential equations of space-time one order. The spinor equations are covariant and are equivalent to Maxwell equations.

Keywords Quantum theory · Spinor equation · Electromagnetic field

1 Introduction

Photons are quantum particles that their behavior is governed by the laws of quantum mechanics. This means, their state are described by wave functions. According to modern quantum field theory, photons, together with all other particles, are the quantum excitations of a field. In the case of photons, these are the excitations of the electromagnetic field. The lowest field excitation of a given type corresponds to one photon and higher field excitations involve more than one photon. This concept of a photon enables one to use the photon wave function not only to describe quantum states of an excitation of the free field but also of the electromagnetic field interacting with a medium [1–12]. Maxwell equations in the matrix Dirac-like form considered during long time by many authors, the interest to the Majorana-Oppenheimer formulation of electrodynamics has grown in recent years [13–20].

After discovering the relativistic equation for a particle with spin 1/2 [21]. In Refs. [8, 9, 15, 20], the authors have proposed to consider the Maxwell theory of electromagnetism as the wave mechanics of the photon, then it must be possible to write Maxwell equations as a Dirac-like equation for a probability quantum wave $\vec{\psi}$, this wave function being expressable by means of the physical E, B fields, and the complex 3-vector wave function satisfying the

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massless Dirac-like equations. Afterwards, much work was done to study spinor and vectors within the Lorentz group theory: Moglich [22], Ivanenko-Landau [23], Neumann [24], van der Waerden [25]. As was shown any quantity which transforms linearly under Lorentz transformations is a spinor. For that reason spinor quantities are considered as fundamental in quantum field theory and basic equations for such quantities should be written in a spinor form. A spinor formulation of Maxwell equations was studied by many authors in [26–30]. In this paper, we give the spinor wave equations of classical electromagnetic field, which are the differential equation of space-time one order. The spinor wave equations are covariant, and the spinors field ψ are corresponding to the reducibility representations of the proper Lorentz group.

2 Spinor Wave Equation of Classical Electromagnetic Field

The differential equation of space-time two order of free electromagnetism wave is

$$\Box A_\mu = 0, \tag{1}$$

and the Lorentz condition is

$$\partial_\mu A_\mu = 0, \tag{2}$$

where

$$\partial_\mu = \left(\nabla, \frac{\partial}{\partial(ict)}\right), \qquad \Box = \partial_\mu \partial_\mu = \nabla^2 - \frac{1}{c^2}\frac{1}{\partial t^2}. \tag{3}$$

Equation (1) has four components A_μ $(\mu = 1, 2, 3, 4)$. Making Eq. (1) into one order differential equation of space-time, the number of field functions should be added.

Defining the function

$$F_{\nu\mu} = \partial_\nu A_\mu, \tag{4}$$

and Eq. (1) becomes

$$\partial_\nu F_{\nu\mu} = \partial_\nu \partial_\nu A_\mu = \Box A_\mu = 0, \tag{5}$$

i.e., the differential equation of space-time one order for free electromagnetism wave is

$$\partial_\nu F_{\nu\mu} = 0. \tag{6}$$

By Eqs. (2) and (4), we have

$$\partial_\mu F_{\nu\mu} = \partial_\nu \partial_\mu A_\mu = 0, \tag{7}$$

with Eqs. (6) and (7), there is

$$\partial_\nu F_{\nu\mu} = \partial_\mu F_{\nu\mu} = \partial_\nu F_{\mu\nu} = 0. \tag{8}$$

From Eq. (8), we have

$$F_{\mu\nu} = F_{\nu\mu}, \tag{9}$$

or

$$F_{\mu\nu} = -F_{\nu\mu}. \tag{10}$$

In classical electrodynamics, the field strength tensor is

$$F^c_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{11}$$

if $F_{\mu\nu} = \partial_\mu A_\nu = F_{\nu\mu} = \partial_\nu A_\mu$ is symmetric, the field strength tensor $F^c_{\mu\nu} = 0$, which isn't reasonable. We should take the antisymmetric form (10), and its matrix form is

$$F_{\mu\nu} = \begin{pmatrix} 0 & F_{12} & F_{13} & F_{14} \\ -F_{12} & 0 & F_{23} & F_{24} \\ -F_{13} & -F_{23} & 0 & F_{34} \\ -F_{14} & -F_{24} & -F_{34} & 0 \end{pmatrix}. \tag{12}$$

For the proper Lorentz group L_p, the irreducibility representations of spin 1 particle or field are D^{10}, D^{01} and $D^{\frac{1}{2}\frac{1}{2}}$, respectively, and the dimension of irreducibility representations are corresponding to three, three and four. Therefore, the reducibility representations of particle or field with spin 1 are

$$D = D^{10} + D^{01}, \tag{13}$$

$$D = D^{10} + D^{01} + D^{\frac{1}{2}\frac{1}{2}}, \tag{14}$$

$$\cdots$$

Equations (13) and (14) are corresponding to six and ten dimensions irreducibility representations, which are the two lowest dimensional irreducibility representations

When $F_{\mu\nu}$ take the antisymmetry tensor (12), which is the representation vector of reducibility representations $D = D^{10} + D^{01}$. Equation (6) can be written as

$$\mu = 1: \quad \partial_1 F_{11} + \partial_2 F_{21} + \partial_3 F_{31} + \partial_4 F_{41} = 0, \tag{15}$$

$$\mu = 2: \quad \partial_1 F_{12} + \partial_2 F_{22} + \partial_3 F_{32} + \partial_4 F_{42} = 0, \tag{16}$$

$$\mu = 3: \quad \partial_1 F_{13} + \partial_2 F_{23} + \partial_3 F_{33} + \partial_4 F_{43} = 0, \tag{17}$$

$$\mu = 4: \quad \partial_1 F_{14} + \partial_2 F_{24} + \partial_3 F_{34} + \partial_4 F_{44} = 0, \tag{18}$$

substituting Eq. (12) into (15)–(18), there is

$$\begin{cases} \partial_2 F_{12} + \partial_3 F_{13} + \partial_4 F_{14} = 0 \\ \partial_1 F_{12} - \partial_3 F_{23} - \partial_4 F_{24} = 0 \\ \partial_1 F_{13} + \partial_2 F_{23} - \partial_4 F_{34} = 0 \\ \partial_1 F_{14} + \partial_2 F_{24} + \partial_3 F_{34} = 0. \end{cases} \tag{19}$$

Equation (19) can be written as the differential form of space-time one order

$$\beta_\mu \partial_\mu \psi = 0 \quad (\mu = 1, 2, 3, 4), \tag{20}$$

where the spinor wave function ψ is

$$\psi = \begin{pmatrix} F_{12} \\ F_{13} \\ F_{14} \\ F_{23} \\ F_{24} \\ F_{34} \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{pmatrix}, \tag{21}$$

and the β matrices are

$$\beta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \beta_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\beta_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \beta_4 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{22}$$

with Eq. (4), we have

$$F^c_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} - F_{\nu\mu} = 2F_{\mu\nu}, \tag{23}$$

the matrix form of $F^c_{\mu\nu}$ and $F_{\mu\nu}$ are

$$F^c_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -\frac{i}{c}E_1 \\ -B_3 & 0 & B_1 & -\frac{i}{c}E_2 \\ B_2 & -B_1 & 0 & -\frac{i}{c}E_3 \\ \frac{i}{c}E_1 & \frac{i}{c}E_2 & \frac{i}{c}E_3 & 0 \end{pmatrix}, \tag{24}$$

and

$$F_{\mu\nu} = \frac{1}{2}F^c_{\mu\nu} = \frac{1}{2}\begin{pmatrix} 0 & B_3 & -B_2 & -\frac{i}{c}E_1 \\ -B_3 & 0 & B_1 & -\frac{i}{c}E_2 \\ B_2 & -B_1 & 0 & -\frac{i}{c}E_3 \\ \frac{i}{c}E_1 & \frac{i}{c}E_2 & \frac{i}{c}E_3 & 0 \end{pmatrix}, \tag{25}$$

the spinor wave function ψ can be written as

$$\psi=\begin{pmatrix}F_{12}\\F_{13}\\F_{14}\\F_{23}\\F_{24}\\F_{34}\end{pmatrix}=\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\\\psi_5\\\psi_6\end{pmatrix}=\begin{pmatrix}\frac{1}{2}B_3\\-\frac{1}{2}B_2\\-\frac{i}{2c}E_1\\\frac{1}{2}B_1\\-\frac{i}{2c}E_2\\-\frac{i}{2c}E_3\end{pmatrix}. \tag{26}$$

In the following, we should give the Spinor form of electromagnetic field for the irreducibility representations D^{10} of the proper Lorentz group. In the natural unit ($c=1$), the Maxwell equations in vacuum are:

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}, \tag{27}$$

$$\nabla\times\vec{B}=\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}=\frac{\partial\vec{E}}{\partial t}, \tag{28}$$

$$\nabla\cdot\vec{E}=0, \tag{29}$$

$$\nabla\cdot\vec{B}=0. \tag{30}$$

With Eqs. (27) and (28), we have

$$\frac{\partial}{\partial t}\frac{1}{\sqrt{2}}\big(\vec{E}+i\vec{B}\big)=-i\nabla\times\frac{1}{\sqrt{2}}\big(\vec{E}+i\vec{B}\big), \tag{31}$$

defining

$$\vec{\psi}=\frac{1}{\sqrt{2}}\big(\vec{E}+i\vec{B}\big), \tag{32}$$

substituting Eq. (32) into (31), there is

$$\frac{\partial\vec{\psi}}{\partial t}=-i\nabla\times\vec{\psi}, \tag{33}$$

the component form of Eq. (33) is

$$\begin{cases}\frac{\partial}{\partial t}(E_x+iB_x)=-i(\frac{\partial}{\partial y}(E_z+iB_z)-\frac{\partial}{\partial z}(E_y+iB_y))\\\frac{\partial}{\partial t}(E_y+iB_y)=i(\frac{\partial}{\partial z}(E_z+iB_z)-\frac{\partial}{\partial z}(E_x+iB_x))\\\frac{\partial}{\partial t}(E_z+iB_z)=-i(\frac{\partial}{\partial x}(E_y+iB_y)-\frac{\partial}{\partial y}(E_x+iB_x)),\end{cases} \tag{34}$$

defining spinor wave function

$$\psi=\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}E_x+iB_x\\E_y+iB_y\\E_z+iB_z\end{pmatrix}. \tag{35}$$

Equation (34) becomes

$$\begin{cases} \frac{\partial}{\partial t}\psi_1 + i\frac{\partial}{\partial y}\psi_3 - i\frac{\partial}{\partial z}\psi_2 = 0 \\ \frac{\partial}{\partial t}\psi_2 - i\frac{\partial}{\partial x}\psi_3 + i\frac{\partial}{\partial z}\psi_1 = 0 \\ \frac{\partial}{\partial t}\psi_3 + i\frac{\partial}{\partial x}\psi_2 - i\frac{\partial}{\partial y}\psi_1 = 0. \end{cases} \tag{36}$$

Equations (29) and (30) can be written as

$$\nabla \cdot (\vec{E} + i\vec{B}) = 0, \tag{37}$$

or

$$\frac{\partial}{\partial x}\psi_1 + \frac{\partial}{\partial y}\psi_2 + \frac{\partial}{\partial z}\psi_3 = 0. \tag{38}$$

Equation (36) can be written as the differential form of space-time one order, i.e., the spinor form

$$\beta^\mu \partial_\mu \psi = 0, \tag{39}$$

i.e.,

$$\left(\beta^0\partial_0 + \beta^1\partial_1 + \beta^2\partial_2 + \beta^3\partial_3\right)\begin{pmatrix}\psi_1 \\ \psi_2 \\ \psi_3\end{pmatrix} = 0, \tag{40}$$

where

$$\begin{aligned} \beta^0 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \beta^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\ \beta^2 &= \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \beta^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{41}$$

The β matrices are hermitian

$$\left(\beta^\mu\right)^+ = \beta^\mu \quad (\mu = 0, 1, 2, 3). \tag{42}$$

Equation (39) conjugation equation is

$$\partial_\mu \psi^+ \beta^\mu = 0, \tag{43}$$

Eq. (40) is

$$\beta^0 \cdot \frac{\partial}{\partial t}\psi = -\vec{\beta} \cdot \nabla\psi, \tag{44}$$

and its conjugation form is

$$\left(\frac{\partial}{\partial t}\psi^+\right)\beta^0 = -\vec{\nabla}\psi^+ \cdot \vec{\beta}. \tag{45}$$

From Eqs. (44) and (45), we have

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \vec{j} = 0, \tag{46}$$

where

$$\rho = \psi^{+}\psi, \qquad \vec{j} = \psi^{+}\vec{\beta}\psi, \tag{47}$$

defining four-dimensional current vector, it is

$$J^{\mu} = \psi^{+}\beta^{\mu}\psi, \tag{48}$$

where

$$j^{0} = \psi^{+}\beta^{0}\psi = \psi^{+}\psi = \frac{1}{2}\big(E^2 + B^2\big), \tag{49}$$

$$j^{i} = \psi^{+}\beta^{i}\psi \qquad (i = 1, 2, 3), \tag{50}$$

the three component of j^i are

$$\begin{aligned} j^{1} &= \psi^{+}\beta^{1}\psi = \begin{pmatrix}\psi_1^* & \psi_2^* & \psi_3^*\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} \\ &= -E_z B_y + E_y B_z = (E \times B)_x, \end{aligned} \tag{51}$$

$$j^{2} = (E \times B)_y, \tag{52}$$

and

$$j^{3} = (E \times B)_z. \tag{53}$$

We find j^0 and $\vec{j} = \vec{E} \times \vec{B}$ are energy and momentum density of electromagnetic field, respectively.

We take the Lagrangian density of spinor Eqs. (39) and (43) as

$$L = \psi^{+}\beta^{\mu}\partial_{\mu}\psi, \tag{54}$$

we have

$$\frac{\partial L}{\partial \psi} = 0, \tag{55}$$

and

$$\frac{\partial L}{\partial(\partial_{\mu}\psi)} = \psi^{+}\beta^{\mu}, \tag{56}$$

substituting Eqs. (55) and (56) into Lagrangian equation

$$\frac{\partial L}{\partial \psi} - \partial_{\mu}\frac{\partial L}{\partial(\partial_{\mu}\psi)} = 0, \tag{57}$$

we get Eq. (43)

$$\partial_\mu \psi^+ \beta^\mu = 0, \tag{58}$$

since

$$\frac{\partial L}{\partial \psi^+} = \beta^\mu \partial_\mu \psi, \tag{59}$$

and

$$\frac{\partial L}{\partial(\partial_\mu \psi^+)} = 0, \tag{60}$$

by Lagrangian equation

$$\frac{\partial L}{\partial \psi^+} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \psi^+)} = 0, \tag{61}$$

we get Eq. (39)

$$\beta^\mu \partial_\mu \psi = 0. \tag{62}$$

We can find the Lagrangian density (54) is the Lagrangian density of spinor electromagnetic field of irreducibility.

3 The Covariance of Spinor Wave Equation

The relativistic quantum theory and quantum field theory should be covariant, the spinor wave Eqs. (20) and (39) should be also.

At Lorentz transformation

$$x'_\mu = \alpha_{\mu\nu} x_\nu, \tag{63}$$

to get

$$x_\mu = \alpha_{\nu\mu} x'_\nu \tag{64}$$

and

$$\partial_\mu = \alpha_{\nu\mu} \partial'_\nu, \tag{65}$$

substituting Eq. (65) into (20) and (39), there is

$$\alpha_{\nu\mu} \beta_\mu \cdot \partial'_\nu \psi = 0, \tag{66}$$

defining

$$\alpha_{\nu\mu} \beta_\mu = L^{-1} \beta_\nu L, \tag{67}$$

i.e.,

$$L \alpha_{\nu\mu} \beta_\mu L^{-1} = \beta_\nu. \tag{68}$$

Equation (66) becomes

$$L^{-1}\beta_\nu \partial'_\nu L\psi = 0, \tag{69}$$

defining

$$\psi'(x') = L\psi. \tag{70}$$

Equation (69) becomes

$$\beta_\nu \partial'_\nu \psi'(x') = 0. \tag{71}$$

The covariance of Eqs. (20) and (39) are proved, and we can give the transformation L.

For a infinitesimal Lorentz transformation

$$x_\mu \to x'_\mu = (\delta_{\mu\nu} + \varepsilon_{\mu\nu})x_\nu, \tag{72}$$

where

$$\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}, \quad |\varepsilon_{\mu\nu}| \ll 1, \tag{73}$$

writing Eq. (72) with infinitesimal operator, it is

$$x'_\mu = \left(\delta_{\mu\nu} + \frac{1}{2}\varepsilon_{\rho\sigma} I^{\mu\nu}_{\rho\sigma}\right)x_\nu, \tag{74}$$

comparing Eq. (72) with (74), there is

$$\frac{1}{2}\varepsilon_{\rho\sigma} I^{\mu\nu}_{\rho\sigma} = \varepsilon_{\mu\nu}, \tag{75}$$

under the infinitesimal Lorentz transformation, the spinor infinitesimal transformation is

$$\psi'(x') = \left(1 + \frac{1}{2}\varepsilon_{\rho\sigma} I_{\rho\sigma}\right)\psi(x) \tag{76}$$

where $I_{\rho\sigma}$ is the infinitesimal operator (matrix) of Lorentz group.

Comparing Eq. (70) with (76), we obtain the transformation matrix L

$$L = 1 + \frac{1}{2}\varepsilon_{\rho\sigma} I_{\rho\sigma} \tag{77}$$

and

$$L^{-1} = 1 - \frac{1}{2}\varepsilon_{\rho\sigma} I_{\rho\sigma}, \tag{78}$$

substituting Eqs. (77) and (78) into (68), there is

$$\left(1 + \frac{1}{2}\varepsilon_{\rho\sigma} I_{\rho\sigma}\right)(\delta_{\nu\mu} + \varepsilon_{\nu\mu})\beta_\mu \left(1 - \frac{1}{2}\varepsilon_{\rho\sigma} I_{\rho\sigma}\right) = \beta_\nu, \tag{79}$$

expanding Eq. (79) to one order of ε, we obtain

$$\frac{1}{2}\varepsilon_{\rho\sigma}(I_{\rho\sigma}\beta_\nu - \beta_\nu I_{\rho\sigma}) = -\varepsilon_{\nu\mu}\beta_\mu. \tag{80}$$

Equation (80) gives the relation between the infinitesimal operator $I_{\rho\sigma}$ and matrix β_ν, i.e., the transformation L is existential. The spinor wave Eqs. (20) and (39) are covariant.

For a infinitesimal Lorentz transformation (72), the solution of Eq. (67) is

$$L = 1 + \frac{1}{2}\epsilon_{\rho\sigma}(\beta_\rho\beta_\sigma - \beta_\sigma\beta_\rho), \tag{81}$$

and

$$L^{-1} = 1 - \frac{1}{2}\epsilon_{\rho\sigma}(\beta_\rho\beta_\sigma - \beta_\sigma\beta_\rho), \tag{82}$$

the infinitesimal operator $I_{\rho\sigma}$ is

$$I_{\rho\sigma} = \beta_\rho\beta_\sigma - \beta_\sigma\beta_\rho, \tag{83}$$

and spin operator is

$$s_3 = \frac{\hbar}{i}I_{12} = \frac{\hbar}{i}(\beta_1\beta_2 - \beta_2\beta_1). \tag{84}$$

Substituting β matrices (41), we can calculate the spin of the irreducibility representations, it is

$$\begin{aligned} s_3 &= \frac{\hbar}{i}(\beta_1\beta_2 - \beta_2\beta_1) \\ &= \frac{\hbar}{i}\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\right) \\ &= \frac{\hbar}{i}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{85}$$

the eigenvalue equation of spin operator is

$$\frac{\hbar}{i}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \lambda\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \tag{86}$$

the equation eigenvalues are

$$\lambda_1 = 0, \qquad \lambda_2 = \hbar, \qquad \lambda_3 = -\hbar, \tag{87}$$

substituting β matrices (22), we can calculate the spin of the reducibility representations, it is

$$\begin{aligned} s_3 &= \frac{\hbar}{i}I_{12} = \frac{\hbar}{i}(\beta_1\beta_2 - \beta_2\beta_1) \\ &= \frac{\hbar}{i}\left(\begin{pmatrix} 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}\begin{pmatrix} 1&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}\right. \end{aligned}$$

$$
-\begin{pmatrix} 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}\Bigg)
$$

$$
=\frac{\hbar}{i}\begin{pmatrix} 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}, \tag{88}
$$

the eigenvalue equation of spin operator is

$$
\frac{\hbar}{i}\begin{pmatrix} 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}\begin{pmatrix} \psi_1\\ \psi_2\\ \psi_3\\ \psi_4\\ \psi_5\\ \psi_6 \end{pmatrix}=\lambda\begin{pmatrix} \psi_1\\ \psi_2\\ \psi_3\\ \psi_4\\ \psi_5\\ \psi_6 \end{pmatrix} \tag{89}
$$

their eigenvalues are

$$
\lambda_1=0,\qquad \lambda_2=0,\qquad \lambda_3=0,\qquad \lambda_4=0,\qquad \lambda_5=\hbar,\qquad \lambda_6=-\hbar. \tag{90}
$$

In the quantization of spinor field ψ, we should find the eigenstates of $\lambda=0$ haven't contribution to the energy and momentum, and the eigenstate of eigenvalue $\lambda=\hbar, -\hbar$ have contribution to the energy and momentum for the irreducibility and reducibility spinor field ψ. So, Eqs. (20) and (39) are the spinor field equations of spin $s=1$, and mass $m=0$, i.e., they are the spinor equations of electromagnetic field.

4 Conclusion

In classical electromagnetism theory, the 4-vector potential A_μ satisfies the differential equation of space-time two order, i.e., d' Alembert equation. In this paper, we give the spinor wave equations of free electromagnetic field, which are the differential equation of space-time one order. The spinor wave equations are covariant and equivalent to Maxwell equations. The spinor wave equations of electromagnetic field can be quantized more easily.

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