Quantum Inf Process (2014) 13:273–282
DOI 10.1007/s11128-013-0648-z

The topological basis realization and the corresponding Heisenberg model of spin-1 chain

Taotao Hu · Hang Ren · Kang Xue

Received: 4 June 2013 / Accepted: 23 September 2013 / Published online: 19 October 2013

Abstract In this paper, it is shown that the Heisenberg model of spin-1 chain can be constructed from the Birman–Wenzl algebra generator while we have got that the Heisenberg model of spin-$\frac{1}{2}$ chain can be constructed from the Temperley–Lieb algebra generator in our previous work (Sun et al. in EPL 94:50001, 2011). Here, we investigate the topological space, we find that the number of topological basis states raise from the previous two to three, and they are also the three eigenstates of a closed four-qubit Heisenberg model of spin-1 chain. Specifically, all the topological basis states are also the spin single states and one of them is the energy single state of the system. It is worth noting that all conclusions we get in this paper are consistent with our previous work (Sun et al. in EPL 94:50001, 2011). These just indicate that the topological basis states have particular properties in the system.

Keywords Yang–Baxter equation · Knot theory · Topological basis · Quantum spin models

T. Hu (✉) · K. Xue
School of Physics, Northeast Normal University, 130024 Changchun, People's Republic of China
e-mail: hutt262@nenu.edu.cn

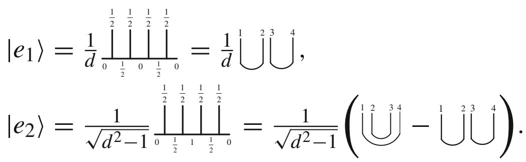
K. Xue
e-mail: Xuekang@nenu.edu.cn

H. Ren
Key Laboratory of airborne Optical Imaging and Measurement, Changchun Institute of Optics, Fine Mechanics and Physics, Chinese Academy of Sciences, 130033 Changchun, People's Republic of China

1 Introduction

In the integrable quantum spin systems, the one-dimensional Heisenberg model under the periodic boundary conditions is one of the fundamental models. In 1931, it was originally introduced by Bethe [2] for the purpose of solving the isotropic Heisenberg spin chain, the Bethe Ansatz, and has been proven an invaluable tool in the field of exactly solved models because of its numerous generalizations and refinements. Variety of physical models and problems [3,4] has also illustrated the versatility of this method and proved it useful. In ref. [5], entanglement control in an anisotropic two-qubit Heisenberg XYZ model with external magnetic fields has been explored, and ref. [6] has investigated the evolution equation of entanglement for bipartite systems. In statistical mechanics, the Temperley–Lieb algebra (TLA) first appeared as a tool to analyze various interrelated lattice models [7], and it was related to link and knot invariants [8]. Up to now, there have been many more models based on the TLA representations, such as the graph models [9–11], the RSOS models [12], and certain vertex models [13,14]. The Temperley–Lieb equivalence has also been naturally extends to the corresponding quantum counterparts of the above statistical mechanical models, such as the quantum RSOS models and the quantum spin-S chains [13,15–19], and it is worth noting that all of which are related to the spin-$\frac{1}{2}$ Heisenberg chain with partial anisotropy (XXZ chain). Recently, our team have also got that the Heisenberg model of spin-$\frac{1}{2}$ chain can be constructed from the TLA generator [1]. However, for spin-1 chain, we need to combine TLA with braid group algebra (BGA), the combination of these two algebra is just the Birman–Wenzl algebra (BWA) [20,21].

In the topological quantum computation theory, based on the $\nu = \frac{5}{2}$ fractional quantum Hall effect (FQHE) [22–25], reference [26,27] investigated the two-dimensional (2D) braid behavior under the exchange of anyons. The orthogonal topological basis states have the form as [22–25],



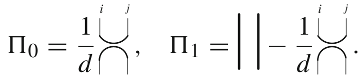
(1)

where the parameter d represents the value of the unknotted loop. At each trivalent vertex, in the middle fusion chains (called conformal block), the internal edges obey the fusion rules as follows,

$$\frac{1}{2} \times \frac{1}{2} = 0 + 1, \quad \frac{1}{2} \times 1 = \frac{1}{2}, \quad 1 \times 1 = 0, \tag{2}$$

$$0 \times 0 = 0, \quad 0 \times \frac{1}{2} = \frac{1}{2}, \quad 0 \times 1 = 1. \tag{3}$$

It is worth pointing out that there are two different fusion channels for two $\frac{1}{2}$ anyons. On the right-hand sides, from the conformal basis to the Staffman graph, Jones–Wenzl projector operators have been applied, i.e.,

 (4)

Recently, many work have shown that topological basis has some important physical applications in topological quantum computation, quantum reexportation, and quantum entanglement [1,28,29]. In ref. [28], based on the topological basis, authors nests the TLA into the four-dimensional (4D) Yang–Baxter equation (YE) and reduces it to the 2D YE, and then they noted that the YE can be tested in terms of quantum optics. Reference [1] connected the topological basis states with a Heisenberg spin-$\frac{1}{2}$ chain. On the other hand, ref. [30] presented an experimental results for a small-scale approximate evaluation of the Jones polynomial via nuclear magnetic resonance (NM). They could obtain the value of the Jones polynomial by measuring the nuclear spin state of the molecule.

Our aim in this work is to connect more topological basis states with a Heisenberg model of spin-1 chain and also study some particular properties of the topological basis states in this system. This paper is organized as follows: in the second section, we recall the TLA, BGA, BWA, and get three new topological basis. Through acting on the subspaces, we get the new nontrivial three-dimensional (3D) Temperley–Lieb matrix representations and braid group matrix representations, which satisfy the 3D TLA and 3D BGA, respectively. In the third section, we show that the Heisenberg model of spin-1 chain can be constructed from the BWA generator. Then, in the topological space, we present a graphic method of constructing the exact solutions for a closed four-qubit Heisenberg model of spin-1 chain. It is interesting that the topological basis states we get are also the three eigenstates of the system. Specifically, all the topological basis states are also the spin single states and one of them is the energy single state of the system.

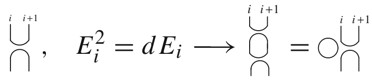
2 TLA, BGA, BWA, and topological basis

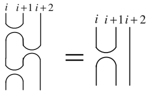
In order to keep the paper self-contained, we first briefly review the theory of the TLA [7]. For each natural number m, the TLA $TL_m(d)$ is generated by $\{I, E_1, E_2, \ldots, E_{m-1}\}$ with the TLA relations:

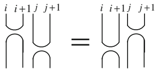
$$\begin{cases} E_i^2 = dE_i & 1 \le i \le m-1 \\ E_i E_{i\pm1} E_i = E_i & 1 \le i \le m \\ E_i E_j = E_j E_i & \mid i - j \mid \ge 2, \end{cases} \tag{5}$$

where d is the unknotted loop $\bigcirc$ in the knot theory, which does not depend on the sites of the lattices. The notation $E_i \equiv E_{i,i+1}$ is used. The E_i represents $1_1 \otimes 1_2 \otimes 1_3 \otimes \cdots \otimes 1_{i-1} \otimes E \otimes 1_{i+2} \cdots 1_m$, and 1_j represents the unit matrix in the jth space V_j. In addition, the TLA is easily understood in terms of knot diagrams in Ref. [30–32]. Using Staffman's graphs, it can be expressed as

$$\begin{cases} E_i \longrightarrow \quad , \quad E_i^2 = dE_i \longrightarrow \\ E_i E_{i\pm1} E_i = E_i \longrightarrow \\ E_i E_j = E_j E_i \longrightarrow \end{cases} \tag{6}$$





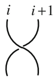


Then, we review the theory of braid groups. Let S_n denotes the braid group on n strands. S_n is generated by elementary braids $\{S_1, S_2, \ldots, S_{n-1}\}$ with the braid relations as follows:

$$\begin{cases} S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1} & 1 \le i < n-2 \\ S_i S_j = S_j S_i & |i-j| \ge 2 \end{cases} \tag{7}$$

where the notation $S_i \equiv S_{i,i+1}$ is used, $S_{i,i+1}$ represents $1_1 \otimes 1_2 \otimes 1_3 \otimes \cdots \otimes 1_{i-1} \otimes S \otimes 1_{i+2} \cdots 1_m$, and 1_j is the unit matrix of the jth particle. Using Staffman's graphs [32], S_i can be expressed as

$$S_i \longrightarrow \tag{8}$$



The BWA is generated by the unit I, the braid operators S_i and the operators E_i with the BWA relations as follows [33,34],

$$\begin{aligned} S_i - S_i^- &= \omega(I - E_i), \\ S_i S_{i+1} S_i &= S_{i+1} S_i S_{i+1}; \quad S_i S_j = S_j S_i, \quad |i-j| \ge 2 \\ E_i E_{i+1} &= E_{i+1} E_i; \ \ E_i E_j = E_j E_i, \quad |i-j| \ge 2 \\ E_i S_i &= S_i E_i = \sigma E_i; \\ S_{i+1} S_i E_{i+1} &= E_i S_{i+1} S_i = E_i E_{i+1}, \\ S_{i+1} E_i S_{i+1} &= S_i^- E_{i+1} S_i^-, \\ E_{i+1} E_i S_{i+1} &= E_{i+1} S_i^-; \ \ S_{i+1} E_i E_{i+1} = S_i^- E_{i+1}, \\ E_i S_{i+1} E_i &= \sigma^- E_i; \\ E_i^2 &= \left(I - \frac{\sigma - \sigma^-}{\omega} \right) E_i \end{aligned} \tag{9}$$

where σ and ω are two independent parameters. The 9×9 Hermitian matrix E with $d = q + 1 + q^-$, which satisfies the TLA relations in Eq. (5), and 9×9 Hermitian matrix S, which satisfies the braid relations in Eq. (7), respectively, take the following form,

$$E = \text{blockdiag}\left(E_1^{(1)}, E_2^{(2)}, E_3^{(3)}, E_2^{(4)}, E_1^{(5)} \right) \tag{10}$$

with the block submarines,

$$E_1^{(1)} = E_1^{(5)} = 0; \quad E_2^{(2)} = E_2^{(4)} = 0,$$

$$E_3^{(3)} = \begin{pmatrix} q & q^{\frac{1}{2}} & 1 \\ q^{\frac{1}{2}} & 1 & q^{-\frac{1}{2}} \\ 1 & q^{-\frac{1}{2}} & q^{-1} \end{pmatrix}, \tag{11}$$

and

$$S = \text{blockdiag}\left(S_1^{(1)}, S_2^{(2)}, S_3^{(3)}, S_2^{(4)}, S_1^{(5)}\right), \tag{12}$$

with the block submarines,

$$S_1^{(1)} = S_1^{(5)} = q, \quad S_2^{(2)} = S_2^{(4)} = \begin{pmatrix} 0 & 1 \\ 1 & q - q^{-1} \end{pmatrix}$$

$$S_3^{(3)} = \begin{pmatrix} 0 & 0 & q^{-1} \\ 0 & 1 & -\omega q^{-\frac{1}{2}} \\ q^{-1} & -\omega q^{-\frac{1}{2}} & \omega(1 - q^{-1}) \end{pmatrix} \tag{13}$$

Here, block diag represents the matrix in block form. The form is $E = \bigoplus_{i=1}^{5} E_k^{(i)} (E_k^{(i)}$ is $k \times k$ matrix), So do the latter matrices, and $\omega = q - q^{-1}$.

In this paper, for the purpose of constructing the Heisenberg model from the BWA operator, we choose $q = 1$ and $d = q + 1 + q^{-} = 3$. So the orthogonal topological basis states in Eq. (1) have the form

$|e_1\rangle = \frac{1}{3}$  ,

$|e_2\rangle = \frac{1}{2\sqrt{2}}\Big($  $- \frac{1}{3}$  $\Big)$. (14)

For BWA, we also get another new topological basis state as follows,

$|e_3\rangle = \frac{\sqrt{30}}{15}\Big($  $- \frac{1}{4}$  $- \frac{1}{4}$  $\Big)$. (15)

Here, the new topological graph  comes from the BGA in the BWA, it means that four spaces 1, 2, 3, and 4 cross-connect. So we get the topological subspaces for the BWA, the topological basis states read

$|e_1\rangle = \frac{1}{3}$  ,

$|e_2\rangle = \frac{1}{2\sqrt{2}}\Big($  $- \frac{1}{3}$  $\Big)$,

$|e_3\rangle = \frac{\sqrt{30}}{15}\Big($  $- \frac{1}{4}$  $- \frac{1}{4}$  $\Big)$. (16)

Acting the operator E of Eq. (11) and S of Eq. (13), respectively, on the topological subspace spanned by $|e_1\rangle$, $|e_2\rangle$, and $|e_3\rangle$, we can get the 3D representations of the operator E and S:

$$E_A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_B = \begin{pmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{8}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$E_C = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_D = \begin{pmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{8}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{17}$$

and

$$S_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{\sqrt{15}}{4} \\ 0 & \frac{\sqrt{15}}{4} & -\frac{1}{4} \end{pmatrix}, \quad S_B = \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{6} & \frac{\sqrt{30}}{6} \\ \frac{\sqrt{2}}{6} & \frac{11}{12} & -\frac{\sqrt{15}}{12} \\ \frac{\sqrt{30}}{6} & -\frac{\sqrt{15}}{12} & -\frac{1}{4} \end{pmatrix},$$
$$S_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{\sqrt{15}}{4} \\ 0 & \frac{\sqrt{15}}{4} & -\frac{1}{4} \end{pmatrix}, \quad S_D = \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{6} & \frac{\sqrt{30}}{6} \\ \frac{\sqrt{2}}{6} & \frac{11}{12} & -\frac{\sqrt{15}}{12} \\ \frac{\sqrt{30}}{6} & -\frac{\sqrt{15}}{12} & -\frac{1}{4} \end{pmatrix} \tag{18}$$

where $(E_A)_{ij} = \langle e_i|E_{12}|e_i\rangle$, $(E_B)_{ij} = \langle e_i|E_{23}|e_i\rangle$, $(E_C)_{ij} = \langle e_i|E_{34}|e_i\rangle$, $(E_D)_{ij} = \langle e_i|E_{41}|e_i\rangle$, so do the matrix S. It is worth noting that these representations in Eqs. (17) and (18) are the new 3D Temperley–Lieb matrix and 3D braid group matrix representations while they all satisfy the 3D TLA and 3D BGA relation. The 3D Temperley–Lieb matrix representations $E_A = E_C$, $E_B = E_D$ and the 3D braid group matrix representations $S_A = S_C$, $S_B = S_D$ also indicate that there is symmetry of exchanging pair indices $12 \leftrightarrow 34$ and $23 \leftrightarrow 41$ for the 3D subspace $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$.

3 The graphic solutions and the topological basis

In ref. [1], authors connected the topological basis states with a Heisenberg spin-$\frac{1}{2}$ chain. In this section, we will show a graphic method of constructing the exact solutions for a four-qubit Heisenberg model of spin-1 chain and then also investigate the particular properties of the topological basis states in this system.

For the ith and $(i+1)$th lattices, the matrix of $(S - E)$ in Eqs. (11) and (13) can be expressed in terms of spin-1 operators

$$\begin{aligned} S_{i,i+1} - E_{i,i+1} &= \frac{1}{2}(L_i^+ L_{i+1}^- + L_i^- L_{i+1}^+) + L_i^3 L_{i+1}^3 \\ &= H_{i,i+1} \end{aligned} \tag{19}$$

where L_i is the spin-1 operator,

$$L_i^3 = |1\rangle_{ii}\langle 1| - |-1\rangle_{ii}\langle -1|,$$
$$L_i^+ = \sqrt{2}(|1\rangle_{ii}\langle 0| + |0\rangle_{ii}\langle -1|) = (L_i^-)^+ \tag{20}$$

and $H_{i,i+1}$ is of the form

$$H_{i,i+1} = \frac{1}{2}(L_i^+ L_{i+1}^- + L_i^- L_{i+1}^+) + L_i^3 L_{i+1}^3 = \vec{L}_i \vec{L}_{i+1}. \tag{21}$$

It is worth noting that the Hamiltonian $H_{i,i+1}$ is just the Heisenberg spin-1 model. So the Heisenberg spin-1 model can be constructed from the BWA generator as follows:

$$H_{i,i+1} = S_{i,i+1} - E_{i,i+1}. \tag{22}$$

Next, we discuss the Hamiltonian of a closed four-qubit Heisenberg model of spin-1 chain under the periodic boundary conditions given by

$$H = J \sum_{i=1}^{4} H_{i,i+1}. \tag{23}$$

where J is the real coupling coefficient. The coupling constant $J < 0$ corresponds to the ferromagnetic case, and $J > 0$ corresponds to the antiferromagnetism case. According to Eq. (22), it is easy to verify that the eigenstates of the Hamiltonian H are the same as the eigenstates of $\Gamma = \sum_{i=1}^{4}(S_{i,i+1} - E_{i,i+1})$. So through combining the above three topological basis states $\{|e_1\rangle, |e_2\rangle, and|e_3\rangle\}$, we can construct the exact solutions for the Hamiltonian H in Eq. (23) as follows:

$$|\Phi_1\rangle = -\frac{\sqrt{5}}{3}|e_1\rangle - \frac{\sqrt{10}}{6}|e_2\rangle + \frac{\sqrt{6}}{6}|e_3\rangle$$
$= \frac{\sqrt{5}}{15}\Big(-\frac{3}{2}$  $-\frac{3}{2}$  $+$  $\Big),$

$|\Phi_2\rangle = \frac{\sqrt{3}}{3}(-|e_1\rangle - \sqrt{2}|e_2\rangle) = \frac{\sqrt{3}}{6}\Big(-$  $+$  $\Big),$

$|\Phi_3\rangle = \frac{1}{3}|e_1\rangle + \frac{\sqrt{2}}{6}|e_1\rangle + \frac{\sqrt{30}}{6}|e_3\rangle = \frac{1}{3}$  (24)

with the corresponding eigenvalues $E_1 = -6J$, $E_2 = -2J$, $E_1 = 0$. As is known, the generators of the Lie algebra of the Heisenberg model are $\vec{I} = \vec{L} = \sum_{i=1}^{4} \vec{l}_i$, where $\vec{I}$ is the total spin-1 operator satisfying $[I_\lambda, I_\mu] = i\epsilon_{\lambda\mu\nu} I_\nu (\lambda, \mu, \nu = 1, 2, 3)$ and $\vec{l}_i$ are the spin-1 operators. It is easy to verify that these eigenstates in Eq. (24) are also the eigenstates of $\vec{L}^2$ (i.e., $\vec{L} = \sum_{i=1}^{4} \vec{l}_i$) and L^z (i.e., $L^z = \sum_{i=1}^{4} l_i^z$). Corresponding to the three eigenstates $\{|\Phi_1\rangle, |\Phi_2\rangle, |\Phi_3\rangle\}$ in Eq. (24), the eigenvalues of $\vec{L}^2$ all are

0, and the eigenvalues of L^z also all are 0. It is worth mentioning that the eigenstates $\{|\Phi_1\rangle, |\Phi_2\rangle, |\Phi_3\rangle\}$ in Eq. (24), which are constructed through the combination of the topological basis states $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$, can also be considered as topological basis states. So the first important point we get is that topological basis states $\{|\Phi_1\rangle, |\Phi_2\rangle, and|\Phi_3\rangle\}$ are the three eigenstates of the closed four-qubit Heisenberg model of spin-1 chain. The second important point is that the three topological basis states all are the spin single states of this system (i.e., the eigenvalues of $\vec{L}^2$ and L^z all are 0), and the number of the topological basis states is equal to the number of the spin single states of the system. The third important point is that the energy single state of the system falls on one of the three topological basis states (i.e., $|\Phi_1\rangle$). Namely, the ground state of the antiferromagnetism system (i.e., $J > 0$) falls on one of the three topological basis states (i.e., $|\Phi_1\rangle$). It is worth noting that all conclusions we get in this paper are consistent with our previous work [1]. These just indicate that the topological basis states have particular properties in the system.

4 Summary

In conclusion, through constructing the Heisenberg model from the BWA operator, we have connected the topological basis states with a Heisenberg model of spin-1 chain and have studied some particular properties of the topological basis states in this system. For BWA, we get a new topological basis state $|e_3\rangle$, and it is shown that the three topological basis states are the three eigenstates of a closed four-qubit Heisenberg model of spin-1 chain. Specifically, the three topological basis states all are the spin single states of the system, and the number of the topological basis states is equal to the number of the spin single states of the system. And we also get that the energy single state of this system falls on one of the three topological basis states. Namely, the ground state of the antiferromagnetism system (i.e., $J > 0$) falls on one of the three topological basis states. It is worth noting that all conclusions we get in this paper are consistent with our previous work [1].

While according to some recent research work [1,28,29], it is known that topological basis has some important physical applications in topological quantum computation, quantum reexportation, and quantum entanglement. In ref. [28], based on the topological basis, authors nests the TLA into the nondirectional (4D) Yang–Baxter equation (YE) and reduces it to the 2D YE, and then they noted that the YE can be tested in terms of quantum optics. Reference [1] connected the topological basis states with a Heisenberg model of spin-1/2 chain. On the other hand, ref. [30] presented an experimental result for a small-scale approximate evaluation of the Jones polynomial via nuclear magnetic resonance (NM). They could obtain the value of the Jones polynomial by measuring the nuclear spin state of the molecule. From these aspects, we could think that the topological basis states have particular properties in the physical system.

However, what we have been discussing in this paper is still an open problem that will require a deal of further investigations. When the number of particle spreads to 2N-qubit ($N = 2, \ldots$) for closed Heisenberg model of spin-1 chain, if all the topological basis states for 2N particles are also the spin single states of the system, if the number of the topological basis states is also equal to the number of the spin single states of the system, and the study of application of more topological basis in quantum information processing, these are work in progress.

Acknowledgments This work was supported by NSF of China (Grant No. 11175043 and Grant No. 11305033) and the Fundamental Research Funds for the Central Universities (Grant No. 11QNJJ012)

References

1. Sun, C.F., Xue, K., Wang, G.C., Zhou, C.C., Du, G.J.: The topological basis realization and the corresponding Heisenberg model spin chain. EPL **94**, 50001 (2011)
2. Bethe, H.: Zur theorie der metalle. Z. Phys. **71**, 205 (1931)
3. Hulthen, L.: Ark. Mat. Astron. Fys. A **26**, 1 (1939)
4. Faddeev, L., Takhtajan, L.: Spectrum and scattering of excitations in the one-dimensional isotropic Heisenberg model. J. Sov. Math. **24**, 241 (1984)
5. Abliz, A., Gao, H.J., Xie, X.C., Wu, Y.S., Liu, W.M.: Entanglement control in an anisotropic two-qubit Heisenberg XYZ model with external magnetic fields. Phys. Rev. A **74**, 052105 (2006)
6. Li, Z.G., Fei, S.M., Wang, Z.D., Liu, W.M.: Evolution equation of entanglement for bipartite systems. Phys. Rev. A **79**, 024303 (2009)
7. Temperley, H.N.V., Lieb, E.H.: Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the 'percolation' problem. Proc. R. Soc. London, Ser. A **322**, 251 (1971)
8. Wadati, M., Deguchi, T., Akutsu, Y.: Exactly solvable models and knot theory. Phys. Rep. **180**, 247 (1989)
9. Owczarek, A.L., Baxter, R.J.: A class of interaction-round-a-face models and its equivalence with an ice-type model. J. Stat. Phys. **49**, 1093 (1987)
10. Pasquier, V.: Two-dimensional critical systems labelled by Dynkin diagrams. Nucl. Phys. B **285**, 162 (1987)
11. Pasquier, V.: Lattice derivation of modular invariant partition functions on the torus. J. Phys. A **20**, L1229 (1987)
12. Andrews, G.E., Baxter, R.J., Forrester, P.J.: Eight-vertex SOS model and generalized Rogers–Ramanujan-type identities. J. Stat. Phys. **35**, 193 (1984)
13. Klümper, A.: New results for q-state vertex models and the pure biquadratic spin-1 Hamiltonian. Europhys. Lett. **9**, 815 (1989)
14. Kulish, P.P.: On spin systems related to the Temperley-Lieb algebra. J. Phys. A: Math. Gen. **36**, L489 (2003)
15. Parkinson, J.B.: On the integrability of the S=1 quantum spin chain with pure biquadratic exchange. J. Phys. C: Solid State Phys. **20**, L1029 (1987)
16. Parkinson, J.B.: The S=1 quantum spin chain with pure biquadratic exchange. J. Phys. C: Solid State Phys. **21**, 3793 (1988)
17. Parkinson, J.B.: J. Phys. (Paris), Colloq., **49**(C8), 1413 (1988)
18. Barber, M., Batchelor, M.T.: Spectrum of the biquadratic spin-1 antiferromagnetic chain. Phys. Rev. B **40**, 4621 (1989)
19. Aufgebauer, B., Klumper, A.: Quantum spin chains of Temperley–Lieb type: periodic boundary conditions, spectral multiplicities and finite temperature. J. Stat. Mech. **2010**, P05018 (2010)
20. Akutsu, Y., Deguchi, T., Wadati, M.: The Yang-Baxter relation: a new tool for knot theory. In: Yang, C.N., Ge, M.L. (eds.) Braid Group, Knot Theroy and Statistical Mechanics, p. 151. World Scientific Publ Co Ltd, Singapore (1989)
21. Chen, Y., Ge, M.L., Xue, K.: Yang-Baxterization of braid group representations. Commun. Math. Phys. **136**, 195 (1991)
22. Ardonne, E., Schoutens, K.: Wavefunctions for topological quantum registers. Ann. Phys. (N.Y.) **322**, 201 (2007)
23. Feiguin, A., Trebst, S., Ludwig, A.W.W., Troyer, M., Kitaev, A., Wang, Z.H., Freedman, M.H.: Interacting anyons in topological quantum liquids: the golden chain. Phys. Rev. Lett. **98**, 160409 (2007)

24. Nayak, C., Simon, S.H., Stern, A., Freedman, M., Sarma, S.D.: Non-Abelian anyons and topological quantum computation. Rev. Mod. Phys. **80**, 1083 (2008)
25. Hikami, K.: Skein theory and topological quantum registers: braiding matrices and topological entanglement entropy of non-Abelian quantum Hall states. Ann. Phys. (N.Y.) **323**, 1729 (2008)
26. Wilczek, F.: Magnetic flux, angular momentum, and statistics. Phys. Rev. Lett. **48**, 1144 (1982)
27. Moore, G., Read, N.: Nonabelions in the fractional quantum Hall effect. Nucl. Phys. B **360**, 362 (1991)
28. Hu, S.W., Xue, K., Ge, M.L.: Optical simulation of the Yang–Baxter equation. Phys. Rev. A **78**, 022319 (2008)
29. Wang, G.C., Xue, K., Sun, C.F., Zhou, C.C., Du, G.J.: e-print arXiv:1012.1474v2
30. Marx, R., Fahmy, A., Kauffman, L., Lomonaco, S., Spörl, A., Pomplun, N., Schulte-Herbrüggen, T., Myers, J.M., Glaser, S.J.: Nuclear-magnetic-resonance quantum calculations of the Jones polynomial. Phys. Rev. A **81**, 032319 (2009)
31. Kauffman, L.H., Lomonaco Jr, S.J.: Braiding operators are universal quantum gates. New J. Phys. **6**, 134 (2004)
32. Kauffman, L.H.: Knots in Physics. World Scientific, Singapore (1991)
33. Birman, J., Wenzl, H.: Braids, links and a new algebra. Trans. Am. Math. Soc. **313**, 249 (1989)
34. Ge, M.L., Xue, K.: Trigonometric Yang-Baxterization of coloured R-matrix. J. Phys. A: Math. **26**, 281–291 (1993)