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Analytical method for the transformation of Zernike polynomial coefficients for scaled, rotated, and translated pupils

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Zernike polynomials provide an excellent metric basis for characterizing the wavefront aberrations of human eyes and optical systems. Since the Zernike expansion is dependent on the size, position, and orientation of the pupil in which the function is defined, it is often necessary to transform the Zernike coefficients between different pupils. An analytic method of transforming the Zernike coefficients for scaled, rotated, and translated pupils is proposed in this paper. The normalized coordinate transformation functions between the polar coordinates of the transformed pupil and the Cartesian coordinates of the original pupil are given. Based on the Cartesian and polar representations of Zernike polynomials, the coefficients' transformation matrix can be derived directly and conveniently. The first 36 terms of standard Zernike polynomials are used to validate the proposed method. For different types of transformation, transformation rules of individual Zernike terms are systematically analyzed, revealing how individual terms of the original pupil transform into terms of the transformed pupil. Numerical examples are presented to demonstrate the validity of the proposed method. Further application of the proposed method to the alignment of pupil-decentered off-axis optical systems is discussed.

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1. INTRODUCTION

Zernike polynomials have been widely used in the field of ophthalmic optics and optical engineering [1–3]. Due to their property of orthogonality over a unit circle, Zernike polynomials have emerged as the preferred method to describe the wavefront aberrations of human eyes and optical systems. Moreover, the lower terms of Zernike polynomials can be related to the Seidel aberrations such as defocus, astigmatism, coma, and spherical aberration. Since the Zernike polynomial coefficients can only be calculated over a specific pupil area, the need to transform the coefficients to scaled, rotated, and translated pupils arises.

To test the repeatability of wavefront sensing of the same eye, multiple tests of individual subjects are often taken, and the Zernike coefficients are compared or averaged. Epidemiological studies of aberrations of different subject populations can also be accomplished through statistical comparison of the wavefront expansion coefficients for the group. Due to the variations in the testing conditions and the natural pupil sizes across the population, the eye movements and different pupil sizes should be taken into consideration. The Zernike coefficients need to be transformed to the same pupil area that is of concern. Besides, for some aberration-correcting methods, such as customized contact lenses and laser refractive surgery, the expected improvements are usually limited by the translation and rotation relative to the pupil. The Zernike coefficients also need to be corrected to account for the eye movements.

Zernike polynomials are also used to represent the state of alignment of optical systems during the alignment process. To develop methods of effectively aligning optical systems, Gray *et al.* [4,5] introduced field dependence to the Zernike polynomial coefficients. By using nodal aberration theory (NAT) [6], the Zernike coefficients were expressed as functions of the wave aberration coefficients, the field vector, and the aberration field decenter vectors that locate the centers of the aberration fields of each surface. With this bridge established between optical design and optical testing, the misalignments of each surface were directly related to the Zernike coefficients obtained by interferometry, providing a valuable insight into methods for effectively aligning optical systems. Gu *et al.* [7] presented an alignment method for three-mirror anastigmatic (TMA) telescopes by using the relationship between the wave aberration coefficients and the Zernike coefficients. However, both of them have only considered the case of coaxial optical systems. By introducing a method of transforming the Zernike coefficients to a decentered pupil, these works can be directly extended to pupil-decentered off-axis optical systems.

Transforming the Zernike coefficients between concentric pupils with different sizes has been the topic of many papers [8–12]. As for the rotation and translation, Guirao *et al.* [13] presented analytical expressions that show the impact of translation and rotation on individual Zernike terms. For the case of rotation, owing to the property of invariance under rotation of the Zernike polynomials, the rotation matrix was constructed directly. For the case of translation, the Zernike polynomials at the displaced coordinates were expanded by means of a Taylor expansion. The translation matrix was constructed by taking the first, second, etc., derivatives of the Zernike polynomials. However, due to the property of Taylor expansion, this method could only be applied to the case of small lateral displacement. The first complete theory to analytically transform Zernike coefficients with regard to scaling, translation, and rotation was proposed by Lundström and Unsbo [14]. They developed the methodology of Campbell [15], where the Zernike polynomials and the coefficients were converted to the complex plane. The transformation could be conveniently accomplished in the complex plane, though additional conversion from complex Zernike coefficients back to the standard coefficients was required. Alternatively, Tatulli [16] also presented an analytic method based on the Fourier transform properties of the Zernike polynomials. The coefficients of the transformation matrix were given in terms of integrals of Bessel functions. However, the calculation was dependent on the respective value of the azimuthal frequencies of the corresponding Zernike terms. There were nine possible combinations as presented by the author. The benefit of such analytic methods is that the explicit relationship between the coefficients and different transformation parameters can be obtained, though a somewhat higher initial effort to derive the expressions is required. On the other hand, Bará *et al.* [17] proposed a numerical method based on the coordinate transformations of a given set of sampling points and the generation of corresponding Zernike polynomials. This approach provides a straightforward way of converting aberration coefficients between different pupils. However, it cannot give information on the dependence of the coefficients on different transformation parameters.

In this paper, an analytic method of transforming Zernike polynomial coefficients for scaled, rotated, and translated pupils is proposed. The normalized coordinate transformation functions for arbitrary scaling, rotation, and translation are derived first. The Cartesian representation of Zernike polynomials is used in the original pupil, and the polar representation is used in the transformed pupil. The properties of Zernike polynomials are reviewed, and the derivation of the coefficients' transformation matrix is presented. The first 36 terms of standard Zernike polynomials are used as examples to validate the proposed method. Transformation rules of individual Zernike terms associated with different types of transformation are analyzed. Further application of the proposed method to the alignment of pupil-decentered off-axis optical systems is discussed.

2. TRANSFORMATION OF COORDINATE SYSTEMS

The scaling, rotation, and translation of a pupil can be parameterized by transforming the reference coordinate system. As illustrated in Fig. 1, the original pupil is defined in the XOY coordinate system, and a scaled, rotated, and translated pupil is defined in the $X'O'Y'$ coordinate system. The radius of the original and transformed pupils are R and R', respectively. $\mathbf{D}$ denotes the translation vector defined in XOY. Since the new pupil must be contained in the original pupil, we have $|\mathbf{D}| \le R - R'$. β is the polar angle of $\mathbf{D}$ measured counterclockwise from X-axis, α is the rotation angle measured counterclockwise from the X axis to X' axis. The scaling factor is defined as

$$B = \frac{R'}{R} \qquad (R' \le R). \tag{1}$$

For any arbitrary point P within the transformed pupil, the coordinate of the point can be expressed by the position vector $\mathbf{r}$ in XOY and, equivalently, by $\mathbf{r}'$ in $X'O'Y'$. The relationship between the two coordinates of P with respect to the corresponding coordinate systems can be expressed as

$$\mathbf{r} = \mathbf{R}(\alpha)\mathbf{r}' + \mathbf{D}. \tag{2}$$

As the vector $\mathbf{r}'$ is defined in $X'O'Y'$, which rotates α from XOY, the rotation operator $\mathbf{R}(\alpha)$ is introduced. Expanding Eq. (2) into scalar form, the Cartesian coordinate of point P in XOY can be obtained:

$$\begin{cases} X = |\mathbf{r}|\cos(\varphi) = |\mathbf{r}'|\cos(\theta+\alpha) + |\mathbf{D}|\cos(\beta) \\ Y = |\mathbf{r}|\sin(\varphi) = |\mathbf{r}'|\sin(\theta+\alpha) + |\mathbf{D}|\sin(\beta) \\ |\mathbf{r}| = \sqrt{X^2+Y^2} \end{cases}, \tag{3}$$

where φ is the polar angle of $\mathbf{r}$ referring to XOY, and θ is the polar angle of $\mathbf{r}'$ referring to $X'O'Y'$. Since the Zernike

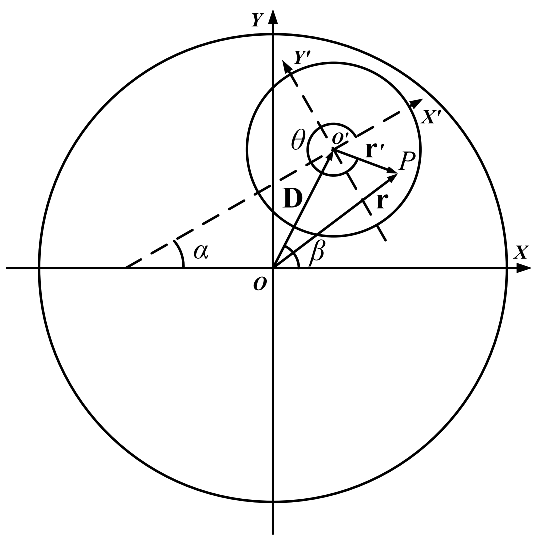

Fig. 1. Illustration of the transformation of reference coordinate system.

polynomials are defined over a unit circle, it is more convenient to express Eq. (3) into a normalized form:

$$\begin{cases} x = \rho\cos(\varphi) = B\rho'\cos(\theta+\alpha) + d\cos(\beta) \\ y = \rho\sin(\varphi) = B\rho'\sin(\theta+\alpha) + d\sin(\beta) \\ \rho = \sqrt{x^2+y^2} \end{cases}, \tag{4}$$

where $d = |\mathbf{D}|/R$ is the normalized translation distance, $\rho = |\mathbf{r}|/R$, and $\rho' = |\mathbf{r}'|/R'$ are defined as the normalized polar radius of point P referring to the two coordinate systems, respectively. Eq. (4) gives the explicit expressions of the coordinate transformation functions between the polar coordinates of the transformed pupil and the Cartesian coordinates of the original pupil. The scaling, rotation, and translation are dealt with simultaneously.

3. TRANSFORMATION OF ZERNIKE COEFFICIENTS

The wavefront aberration over the original pupil defined in XOY can be described by a linear combination of a set of Zernike polynomials:

$$W(\rho,\varphi) = \sum_{j=1}^{M} c_j Z_j(\rho,\varphi), \tag{5}$$

where M is the index of the highest-order Zernike term and c_j is the coefficient of the jth Zernike term. The Zernike polynomials over a unit radius pupil using the amplitude normalization can be defined as

$$Z_j(\rho,\varphi) = Z_n^m(\rho,\varphi) = R_n^m(\rho)\begin{cases} \cos(m\varphi) & \text{for } m \geq 0 \\ \sin(|m|\varphi) & \text{for } m < 0 \end{cases}, \tag{6}$$

where n and m are integers (including zero), $n - |m| \geq 0$ and even. The index n represents the radial degree or the order of the polynomial, since it represents the highest power of ρ in the polynomial, and m is called the azimuthal frequency. $R_n^m(\rho)$ is the radial factor given by

$$R_n^m(\rho) = \sum_{s=0}^{(n-|m|)/2} \frac{(-1)^s (n-s)!}{s!\left(\frac{n-m}{2}-s\right)!\left(\frac{n+m}{2}-s\right)!}\rho^{n-2s}. \tag{7}$$

The norm of a Zernike polynomial term is then given by

$$N_{nm} = \int_0^{2\pi}\int_0^1 Z_n^m(\rho,\varphi)Z_n^m(\rho,\varphi)\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi = \frac{\pi(1+\delta_{0m})}{2(n+1)}, \tag{8}$$

where δ_{0m} is the Kronecker symbol. For the convenience of coordinate transformation as described in Eq. (4), $W(\rho,\varphi)$ needs to be transformed into the Cartesian coordinates

$$W(\rho,\varphi) = W(x,y,\rho) = \sum_{j=1}^{M} c_j Z_j(x,y,\rho). \tag{9}$$

Substituting the following trigonometric identities into Eqs. (6) and (7), the Cartesian representation of the Zernike polynomials can be derived:

$$\begin{cases} \cos m\varphi = \sum_{v=0}^{\lfloor \frac{m}{2}\rfloor}(-1)^v C_m^{2v}\cos^{m-2v}\varphi\sin^{2v}\varphi, & m \geq 0 \\ \sin|m|\varphi = \sum_{v=0}^{\left\lfloor \frac{|m|-1}{2}\right\rfloor}(-1)^v C_{|m|}^{2v+1}\cos^{|m|-2v-1}\varphi\sin^{2v+1}\varphi, & m<0 \\ x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases}. \tag{10}$$

The Cartesian representation of the Zernike polynomials can be written as

$$\begin{aligned} Z_j(x,y,\rho) &= Z_n^m(x,y,\rho) \\ &= \begin{cases} \sum_{s=0}^{\frac{n-m}{2}}\sum_{v=0}^{\lfloor\frac{m}{2}\rfloor}\frac{(-1)^{v+s}(n-s)!C_m^{2v}\rho^{n-m-2s}x^{m-2v}y^{2v}}{s!(\frac{n-m}{2}-s)!(\frac{n+m}{2}-s)!}, & m\geq 0 \\ \sum_{s=0}^{\frac{n-|m|}{2}}\sum_{v=0}^{\left\lfloor\frac{|m|-1}{2}\right\rfloor}\frac{(-1)^{v+s}(n-s)!C_{|m|}^{2v+1}\rho^{n-|m|-2s}x^{|m|-2v-1}y^{2v+1}}{s!(\frac{n-m}{2}-s)!(\frac{n+m}{2}-s)!}, & m<0 \end{cases}. \end{aligned} \tag{11}$$

Similarly, the wavefront aberration over the transformed pupil defined in $X'O'Y'$ can be expressed as

$$W'(\rho',\theta) = \sum_{i=1}^{M'} c_i' Z_i(\rho',\theta), \tag{12}$$

where M' is the index of the highest-order Zernike term. A different set of Zernike polynomials can be used. c_i' is the coefficient of the ith Zernike term. For any arbitrary points in the transformed pupil, the wavefront aberration described by Eqs. (9) and (12), referring to different coordinate systems, are equivalent:

$$W'(\rho',\theta) = W(x,y,\rho) = \sum_{j=1}^{M} c_j Z_j(x,y,\rho). \tag{13}$$

Substituting Eq. (4) into Eq. (13), the wavefront aberration function of the transformed pupil, expressed in terms of ρ' and θ, is obtained. Since the Zernike polynomials are orthogonal and complete over the unit radius circle, the coefficients for a given set of Zernike polynomials of the transformed pupil can be calculated as

$$c_i' = \frac{1}{N_i}\int_0^{2\pi}\int_0^1 W(x,y,\rho)Z_i(\rho',\theta)\rho'\mathrm{d}\rho'\mathrm{d}\theta, \tag{14}$$

where N_i is the norm of $Z_i(\rho',\theta)$, and x, y, and ρ satisfy the description in Eq. (4). Moreover, Eq. (14) can also be expressed in a matrix form as

$$\mathbf{c}' = \mathbf{T}\mathbf{c}, \tag{15}$$

where $\mathbf{c}' = [c_1', c_2', \ldots, c_{M'}']'$ and $\mathbf{c} = [c_1, c_2, \ldots, c_M]'$ are column vectors containing the Zernike coefficients, and $\mathbf{T}$ is a $M' \times M$ matrix defined as the transformation matrix. Eq. (15) indicates that each coefficient of the transformed pupil is simply a linear combination of the original ones. The elements of $\mathbf{T}$ can be calculated as

$$T_{kl} = \frac{1}{N_k}\int_0^{2\pi}\int_0^1 Z_l(x,y,\rho)Z_k(\rho',\theta)\rho'\mathrm{d}\rho'\mathrm{d}\theta, \tag{16}$$

where T_{kl} denotes the element in the kth row and lth column of $\mathbf{T}$.

Based on the Cartesian and polar representations of Zernike polynomials, the transformation matrix can be derived directly. No particular operation on the polynomials is required. Attributed to the orthogonality of Zernike polynomials, the calculation of each coefficient is independent and is only related to the corresponding Zernike term, which allows the transformation of coefficients between different sets of Zernike polynomials. As a result, one can only calculate the coefficients of concerned Zernike terms. However, in the case that the overall wavefront aberration is concerned, enough terms of Zernike polynomials should be considered. Comastri *et al.* [18] have given the selection rules of Zernike terms for scaling and translation and analyzed the missing terms associated with certain translation directions. Lundström and Unsbo [14] also have briefly discussed how the individual Zernike coefficients couple to each other when the pupil is transformed. Generally speaking, the set of Zernike polynomials that are used to describe the wavefront aberration over the transformed pupil should include all terms of radial degrees equal to and smaller than the highest n of the original set.

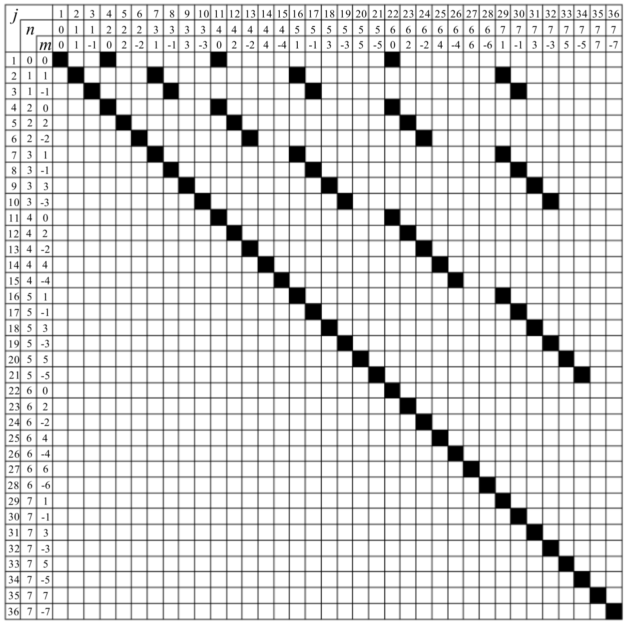

Fig. 2. Scaling matrix of the first 36 terms of standard Zernike polynomials; the filled squares denote the nonzero elements.

4. EXAMPLE

The first 36 terms of standard Zernike polynomials listed in Appendix A are used in this paper to validate the method described in the previous section. Nevertheless, it is worth pointing out that the proposed method can be applied to any given set of Zernike polynomials. By calculating the transformation matrixes of different types of transformation, the transformation rules of individual terms are systematically analyzed. A numerical example of simultaneous scaling, translation, and rotation is presented.

A. Scaling

Concentric scaling of the pupil is considered first. By substituting $\alpha = 0$, $d = 0$, and $\beta = 0$, the transformation matrix for pure scaling of the pupil can be derived. As illustrated in Fig. 2, the scaling matrix is shown in a black-and-white image, where the filled squares denote the nonzero elements.

According to the definition of transformation matrix, the columns of the matrix correspond to the terms of the original Zernike set, and the rows correspond to the terms of the new set. By referring to the radial degree and the azimuthal frequency of each term associated with the nonzero elements, it can be found that the scaling of any Zernike term of radial degree n and azimuthal frequency m gives rise to contributions to terms of the same azimuthal frequency and equal or smaller radial degree, which can be expressed as

$$(n, m) \Rightarrow (n' \leq n, m' = m), \tag{17}$$

where (n, m) and (n', m') denote the radial degree and the azimuthal frequency of the original and new Zernike sets, respectively. For example, the 23rd term of the original set ($n = 7$, $m = 1$) contributes to the 2nd ($n' = 1$, $m' = 1$), 7th ($n' = 3$, $m' = 1$), 14th ($n' = 5$, $m' = 1$), and 23rd ($n' = 7$, $m' = 1$) terms of the new set, while the 24th term ($n = 7$, $m = -1$) contributes to the 3rd ($n' = 1$, $m' = -1$), 8th ($n' = 3$, $m' = -1$), 15th ($n' = 5$, $m' = -1$), and 24th ($n' = 7$, $m' = -1$) terms.

All of the nonzero elements of the matrix are polynomials of the scaling factor B and are only dependent on the radial degree n and n' of the corresponding Zernike terms. Appendix B provides the expressions of all nonzero elements of the scaling matrix up to the 7th order. It is found that our results match exactly with the formula provided by Janssen and Dirksen [12]. According to Eq. (4) in [12], a general expression of the elements can be given as

$$T_{nmn'm'} = \delta_{mm'}[R_n^{n'}(B) - R_n^{n'+2}(B)], \qquad n' \leq n, \tag{18}$$

where $T_{nmn'm'}$ denotes the element associated with the (n, m) and (n', m') Zernike term, and $\delta_{mm'}$ is the Kronecker symbol.

B. Scaling and Translation

The transformed pupil should be contained in the original one, which means the translation vector must be subject to the constraint $|\mathbf{D}| \leq R - R'$. As a result, the scaling and translation must be performed simultaneously. Let the rotation angle $\alpha = 0$, and the transformation matrix of scaling and translation is obtained. As pointed out by previous research [14,18], the couplings between individual Zernike terms are dependent on the direction of the translation. With the help of the derived transformation matrix, this can be explained theoretically. It can be found that the common factors of certain elements of the matrix are $\sin(a\beta)$ or $\cos(a\beta)$ $(a = 1, 2, \ldots, 6, 7)$. For specific translation angles that lead to $\sin(a\beta) = 0$ or $\cos(a\beta) = 0$, the corresponding elements will also be equal to zero. As a result, the associated couplings between individual Zernike terms will be absent. As shown in Fig. 3, the filled squares denote the elements independent from the translation angle, S1–S7 denote the elements of which the common factors are $\sin(\beta) \sim \sin(7\beta)$, and C1–C7 denote the elements of which

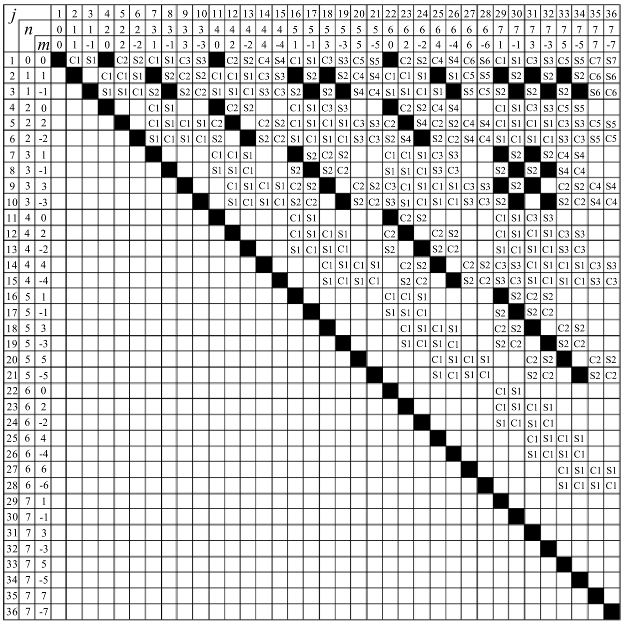

Fig. 3. Scaling and translation matrix of the first 36 terms of standard Zernike polynomials; the filled squares denote the elements independent from the translation angles, S1–S7 denote the elements of which the common factors are $\sin(\beta) \sim \sin(7\beta)$, and C1–C7 denote the elements of which the common factors are $\cos(\beta) \sim \cos(7\beta)$.

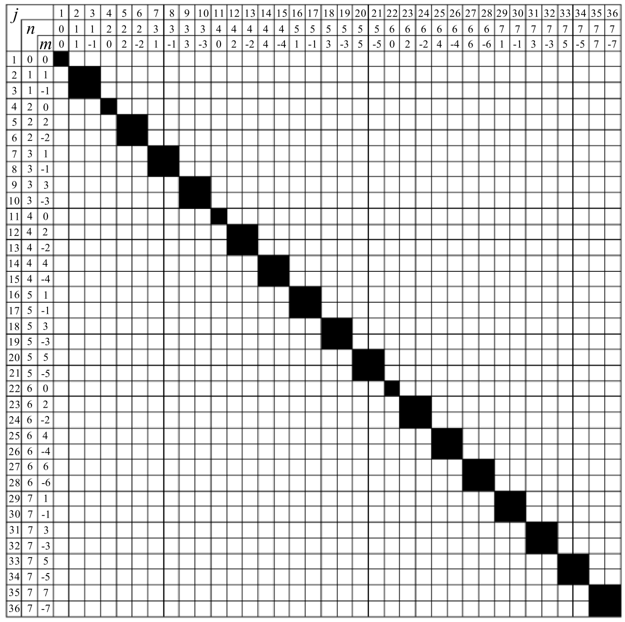

Fig. 4. Rotation matrix of the first 36 terms of standard Zernike polynomials; the black blocks denote the nonzero elements.

the common factors are $\cos(\beta) \sim \cos(7\beta)$. Nevertheless, except for certain translation angles, the general transformation rule of individual terms for scaling and translation can be given as

$$(n, m) \Rightarrow \begin{pmatrix} n' + |m'| \leq n + |m| \\ n' - |m'| \leq n - |m| \\ m' \neq -m \text{ for } m \neq 0 \end{pmatrix} + \left(\begin{array}{c} m' = -m \\ \text{for} \begin{cases} |m| = 1, & n' < n \\ n \text{ even}, & n' < n - 2 \end{cases} \end{array} \right). \quad \textbf{(19)}$$

Generally speaking, the scaling and translation of any Zernike term of (n, m) gives rise to contributions to the same (n, m) term and certain terms of smaller radial degrees.

C. Rotation

Rotation of the pupil can be treated independently from the scaling and translation. By substituting $B = 1$, $d = 0$, and $\beta = 0$, the rotation matrix is obtained and illustrated in Fig. 4. As can be seen from the figure, the rotation of any Zernike term of radial degree n and azimuthal frequency m will give rise to contributions to the terms of $(n, \pm m)$:

$$(n, m) \Rightarrow (n' = n, m' = \pm m). \quad \textbf{(20)}$$

Moreover, it is found that the expression of the elements of the rotation matrix can be given as

$$T_{nmn'm'} = \delta_{nn'} \delta_{|m||m'|} \begin{cases} 1 & \text{for } m' = 0 \\ \cos(m'\alpha) & \text{for } m' = m \\ \sin(m'\alpha) & \text{for } m' = -m \end{cases}. \quad \textbf{(21)}$$

D. COMBINED TRANSFORMATION

Finally, a combined transformation of scaling, translation, and rotation is considered. As the rotation of the pupil can be treated independently from the scaling and translation, the combined transformation matrix can be calculated by multiplying the rotation matrix by the scaling and translation matrix. However, there is actually no need to treat them separately. By using the proposed method, the scaling, translation, and rotation can be performed simultaneously, and the combined transformation matrix can be calculated directly. The transformation matrix is illustrated in Fig. 5. Obviously, the transformation rule of individual terms can be given as

$$(n, m) \Rightarrow \begin{pmatrix} n' + |m'| \leq n + |m| \\ n' - |m'| \leq n - |m| \end{pmatrix}. \quad \textbf{(22)}$$

To sum up, an original Zernike term of radial degree n never affects a term of radial degree higher than n. Therefore, a more practical and quick selection rule for Zernike polynomials can be given: the set of Zernike polynomials that is used to describe the wavefront aberration over the transformed pupil should include all terms of radial degrees equal to and smaller than the highest n of the original set, just as presented in the previous section.

Finally, consider the case of simultaneous scaling ($B = 0.6$), translation (normalized $d = 0.3$ and $\beta = 40°$), and rotation ($\alpha = 60°$). According to the above analysis, the first 36 terms of standard Zernike polynomials are used to describe the wavefront aberrations of both the original and the transformed pupils. By using the proposed method, the transformation matrix is derived, and the new set of coefficients is calculated directly from a given set of Zernike coefficients. Using the Zernike polynomial coefficients, the wavefront maps of the pupils are constructed. The wavefront maps and the Zernike polynomial

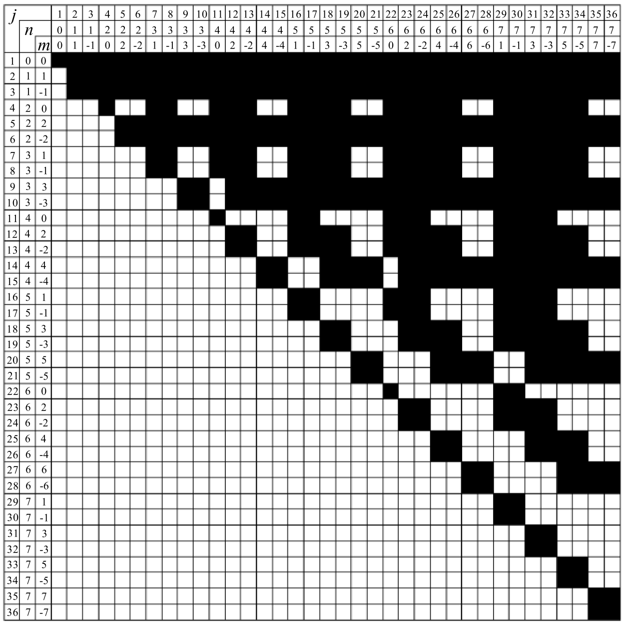

Fig. 5. Combined transformation matrix of the first 36 terms of standard Zernike polynomials; the filled squares denote the nonzero elements.

coefficients of the original and the transformed pupils are shown in Fig. 6. For comparison, the case of scaling and translation [shown in Fig. 6(b)] and the case of scaling, translation, and rotation [shown in Fig. 6(c)] are considered, respectively. Note that the Zernike coefficients of both pupils are calculated directly from the original set. As can be seen from the figure, the calculated wavefront of the transformed pupils matches perfectly with the corresponding area in the original pupil.

5. FURTHER APPLICATION

The alignment of pupil-decentered off-axis optical systems has always been a challenging task in the field of optical engineering. To our knowledge, most existing alignment methods are based on the Zernike polynomial coefficients obtained from interferometric testing. On the other hand, by using NAT [6], analytic expressions of Zernike polynomials of coaxial systems can be derived [4,5], where the field and misalignment dependences are introduced to the Zernike polynomial coefficients. According to NAT, the wavefront aberration expansion of a misaligned coaxial optical system is given as

$$W = \sum_{i}^{\#\text{surface}} W_i = \sum_{i}^{\#\text{surface}} \sum_{p}^{\infty}\sum_{q}^{\infty}\sum_{r}^{\infty} W_{abr,i}(\vec{H}_{Ai}\cdot\vec{H}_{Ai})^p(\vec{\rho}\cdot\vec{\rho})^q(\vec{H}_{Ai}\cdot\vec{\rho})^r$$

$$\vec{H}_{Ai} = \vec{H} - \vec{\sigma}_i, \quad a = 2p + r, \quad b = 2q + r, \tag{23}$$

where the total aberration W is the sum of the individual surface contributions W_i, the subscript i is the surface number, $W_{\text{abr},i}$ is the wave aberration coefficient for surface i, $\vec{\rho}$ is the pupil vector representing the pupil position, $\vec{H}_{Ai}$ is defined

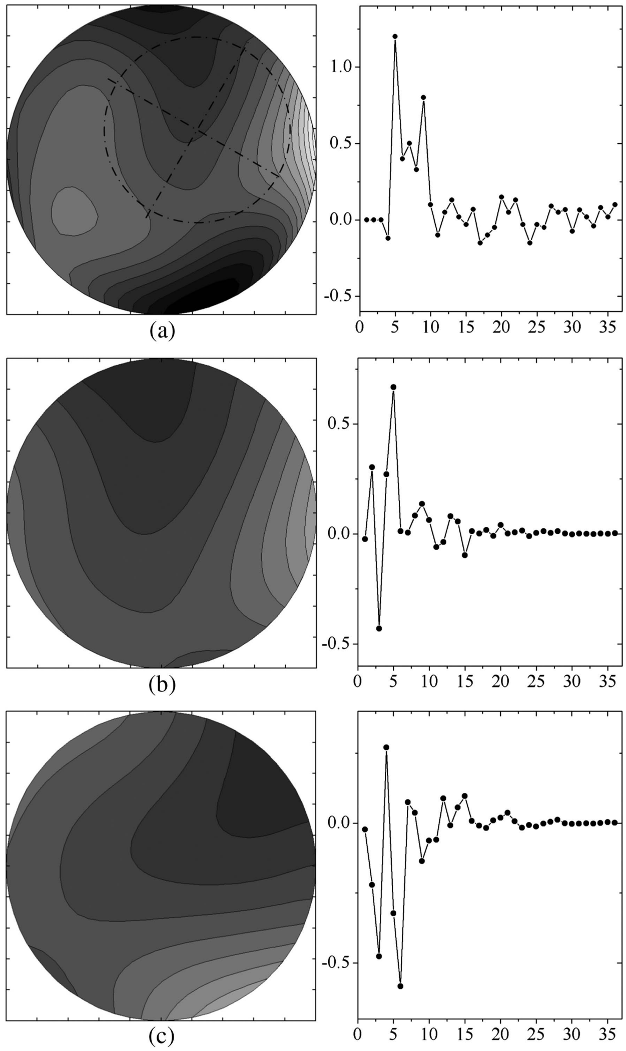

Fig. 6. Wavefront maps and corresponding Zernike polynomial coefficients of the original and transformed pupils. (a) original pupil; (b) scaled ($B = 0.6$) and translated (normalized $d = 0.3$ and $\beta = 40°$) pupil; (c) scaled ($B = 0.6$), translated (normalized $d = 0.3$ and $\beta = 40°$), and rotated ($\alpha = 60°$) pupil. The coefficients of (b) and (c) are both calculated directly from the original set.

as the effective aberration field height of surface i, $\vec{H}$ is the field vector representing the field position, and $\vec{\sigma}_i$ is the aberration field decenter vector that locates the center of the aberration field of surface i and is determined by the misalignment of the surface [7,19]. Substituting Eq. (23) into a similar equation such as Eq. (14), the analytic expressions of field and misalignment dependent Zernike coefficients can be derived:

$$A_j = \sum_{i}^{\#\text{surface}} A_{j,i}(\vec{H},\vec{\sigma}_i) = \sum_{i}^{\#\text{surface}} \frac{1}{N_j}\int_0^{2\pi}\int_0^1 W_i(\vec{\rho},\vec{H},\vec{\sigma}_i)Z_j(\rho,\theta)\rho \mathrm{d}\rho \mathrm{d}\theta, \tag{24}$$

where A_j is the coefficient of $Z_j(\rho,\theta)$ and is the sum of the surface contributions $A_{j,i}$; N_j denotes the norm of $Z_j(\rho,\theta)$.

The Zernike coefficients are expressed as functions of the field vectors and the aberration field decenter vectors. Equating them with the Zernike polynomial coefficients measured on several field positions by interferometric testing,

$$\sum_{i}^{\#\text{surface}} \mathbf{A}_i(\vec{H}, \vec{\sigma}_i) = \mathbf{A}_{\text{measured}}^{\text{co-axis}}, \tag{25}$$

where **A** denotes the column vector containing coefficients of concerned Zernike terms. Generally speaking, the terms that are sensitive to the amounts of misalignment should be used. Given the field vectors and the Zernike coefficients of each measurement, Eq. (25) can be solved. The aberration field decenter vectors and, further, the misalignments (tilt and decenter) of each individual surface can be calculated directly.

Note that Eq. (23) is defined in the coaxial optical systems. As a result, Eq. (25) can only be applied to the alignment of coaxial systems. However, by introducing the proposed method of transforming Zernike coefficients, it can be directly extended to a pupil-decentered off-axis optical system:

$$\mathbf{T} \sum_{i}^{\#\text{surface}} \mathbf{A}_i(\vec{H}, \vec{\sigma}_i) = \mathbf{A}_{\text{measured}}^{\text{off-axis}}, \tag{26}$$

where **T** is the corresponding transformation matrix. In fact, many alignment methods for coaxial optical systems [7,20] can also be directly extended to off-axis optical systems in the same way. Certainly, there are still many practical problems that have to be considered. For example, the effects of optical-surface deformation and measurement errors on the wavefront aberration of the system should be taken into account. The algorithm of solving the overdetermined nonlinear equations also needs to be studied. More issues on this topic will be investigated in detail in our future research and papers.

6. CONCLUSION

An analytic method of transforming the Zernike polynomial coefficients for scaled, translated, and rotated pupils is proposed in this paper. The transformation matrix that accounts for the effects of scaling, translation, and rotation simultaneously can be derived directly and conveniently. The calculation of each element of the matrix is independent and is only related to the corresponding Zernike terms, which allows the transformation of coefficients between different sets of Zernike polynomials and the coefficients of concerned Zernike terms being calculated directly and independently. The 36 terms of standard Zernike polynomials are used as an example to validate the effectiveness of the proposed method. By analyzing the characteristics of the transformation matrixes of different types of transformation, the transformation rules of individual Zernike terms are presented, revealing how individual terms of the original pupil transform into terms of the transformed pupil. The presented transformation rules also provide a valuable insight into relationships of wavefront aberrations of a pupil-decentered off-axis system and its coaxial parent system. Furthermore, a general rule for selecting the set of Zernike polynomials that is used to describe the wavefront aberration over the transformed pupil is given. Finally, numerical examples demonstrate the validity of the proposed method. Appendix C gives the MATLAB code that is used to derive the transformation matrix for the 36 terms of Zernike polynomials. The readers may also see Code 1, Ref. [21] for the explicit expression of the transformation matrix, which can be used directly.

Beside its common application in the field of ophthalmic optics, further application of the proposed method to the alignment of pupil-decentered off-axis optical systems is also introduced. By using the coefficients' transformation matrix, many existing alignment methods for coaxial optical systems can be directly extended to off-axis optical systems. The alignment of off-axis optical systems will be investigated in detail in our future research.

APPENDIX A: FIRST 36 TERMS OF STANDARD ZERNIKE POLYNOMIALS IN CARTESIAN COORDINATES

j	n	m	Zernike Polynomial	Norm
1	0	0	1	π
2	1	1	x	$\pi/4$
3	1	−1	y	$\pi/4$
4	2	0	$2\rho^2 - 1$	$\pi/3$
5	2	2	$x^2 - y^2$	$\pi/6$
6	2	−2	$2xy$	$\pi/6$
7	3	1	$x(3\rho^2 - 2)$	$\pi/8$
8	3	−1	$y(3\rho^2 - 2)$	$\pi/8$
9	3	3	$x(4x^2 - 3\rho^2)$	$\pi/8$
10	3	−3	$y(4x^2 - \rho^2)$	$\pi/8$
11	4	0	$6\rho^4 - 6\rho^2 + 1$	$\pi/5$
12	4	2	$(x^2 - y^2)(4\rho^2 - 3)$	$\pi/10$
13	4	−2	$2xy(4\rho^2 - 3)$	$\pi/10$
14	4	4	$\rho^4 - 8x^2y^2$	$\pi/10$
15	4	−4	$4xy(x^2 - y^2)$	$\pi/10$
16	5	1	$x(10\rho^4 - 12\rho^2 + 3)$	$\pi/12$
17	5	−1	$y(10\rho^4 - 12\rho^2 + 3)$	$\pi/12$
18	5	3	$x(x^2 - 3y^2)(5\rho^2 - 4)$	$\pi/12$
19	5	−3	$y(3x^2 - y^2)(5\rho^2 - 4)$	$\pi/12$
20	5	5	$x(16x^4 - 20x^2\rho^2 + 5\rho^4)$	$\pi/12$
21	5	−5	$y(16x^4 - 12x^2\rho^2 + \rho^4)$	$\pi/12$
22	6	0	$20\rho^6 - 30\rho^4 + 12\rho^2 - 1$	$\pi/7$
23	6	2	$(x^2 - y^2)(15\rho^4 - 20\rho^2 + 6)$	$\pi/14$
24	6	−2	$2xy(15\rho^4 - 20\rho^2 + 6)$	$\pi/14$
25	6	4	$(\rho^4 - 8x^2y^2)(6\rho^2 - 5)$	$\pi/14$
26	6	−4	$4xy(x^2 - y^2)(6\rho^2 - 5)$	$\pi/14$
27	6	6	$32x^6 - 48\rho^2x^4 + 18\rho^4x^2 - \rho^6$	$\pi/14$
28	6	−6	$2xy(16x^4 - 16\rho^2x^2 + 3\rho^4)$	$\pi/14$
29	7	1	$x(35\rho^6 - 60\rho^4 + 30\rho^2 - 4)$	$\pi/16$
30	7	−1	$y(35\rho^6 - 60\rho^4 + 30\rho^2 - 4)$	$\pi/16$
31	7	3	$x(4x^2 - 3\rho^2)(21\rho^4 - 30\rho^2 + 10)$	$\pi/16$

(Table continued)

j	n	m	Zernike Polynomial	Norm
32	7	−3	$y(4x^2 - \rho^2)(21\rho^4 - 30\rho^2 + 10)$	$\pi/16$
33	7	5	$x(7\rho^2 - 6)(5\rho^4 - 20\rho^2 x^2 + 16x^4)$	$\pi/16$
34	7	−5	$y(7\rho^2 - 6)(\rho^4 - 12\rho^2 x^2 + 16x^4)$	$\pi/16$
35	7	7	$x(64x^6 - 112\rho^2 x^4 + 56\rho^4 x^2 - 7\rho^6)$	$\pi/16$
36	7	−7	$y(64x^6 - 80\rho^2 x^4 + 24\rho^4 x^2 - \rho^6)$	$\pi/16$

APPENDIX B: ELEMENTS OF THE SCALING MATRIX

n	n'	Expressions of the Elements
0	0	1
1	1	B
2	0	$B^2 - 1$
2	2	B^2
3	1	$2B^3 - 2B$
3	3	B^3
4	0	$2B^4 - 3B^2 + 1$
4	2	$3B^4 - 3B^2$
4	4	B^4
5	1	$5B^5 - 8B^3 + 3B$
5	3	$4B^5 - 4B^3$
5	5	B^5
6	0	$4B^6 - 10B^4 + 6B^2 - 1$
6	2	$9B^6 - 15B^4 + 6B^2$
6	4	$5B^6 - 5B^4$
6	6	B^6
7	1	$14B^7 - 30B^5 + 20B^3 - 4B$
7	3	$14B^7 - 24B^5 + 10B^3$
7	5	$6B^7 - 6B^5$
7	7	B^7

APPENDIX C: MATLAB CODE

```
%Return symbolic transformation matrix for 36 terms of standard
Zernike polynomials
clear all;
syms B...      %Scaling factor
     d...      %Normalized translation distance
     beta...   %Polar angle of the translation vector
     alpha...  %Rotation angle
     r0...     %Normalized polar radius of the transformed pupil
     theta;    %Polar angle of the transformed pupil
%Coordinates transformation
x=B*r0*cos(theta+alpha)+d*cos(beta);
y=B*r0*sin(theta+alpha)+d*sin(beta);
r=sqrt(x^2+y^2);
%Cartesian representation of 36 terms of standard Zernike
polynomials
Z=[1
   x
   y
   2*r^2-1
   2*x^2-r^2
   2*x*y
   x*(3*r^2-2)
   y*(3*r^2-2)
   4*x^3-3*r^2*x
   -y*(r^2-4*x^2)
   6*r^4-6*r^2+1
```

(Table continued)

```
   -(r^2-2*x^2)*(4*r^2-3)
   2*x*y*(4*r^2-3)
   r^4-8*r^2*x^2+8*x^4
   -4*x*y*(r^2-2*x^2)
   x*(10*r^4-12*r^2+3)
   y*(10*r^4-12*r^2+3)
   -x*(5*r^2-4)*(3*r^2-4*x^2)
   -y*(r^2-4*x^2)*(5*r^2-4)
   5*r^4*x-20*r^2*x^3+16*x^5
   y*(r^4-12*r^2*x^2+16*x^4)
   20*r^6-30*r^4+12*r^2-1
   -(r^2-2*x^2)*(15*r^4-20*r^2+6)
   2*x*y*(15*r^4-20*r^2+6)
   (6*r^2-5)*(r^4-8*r^2*x^2+8*x^4)
   -4*x*y*(r^2-2*x^2)*(6*r^2-5)
   -r^6+18*r^4*x^2-48*r^2*x^4+32*x^6
   2*x*y*(3*r^4-16*r^2*x^2+16*x^4)
   x*(35*r^6-60*r^4+30*r^2-4)
   y*(35*r^6-60*r^4+30*r^2-4)
   -x*(3*r^2-4*x^2)*(21*r^4-30*r^2+10)
   -y*(r^2-4*x^2)*(21*r^4-30*r^2+10)
   x*(7*r^2-6)*(5*r^4-20*r^2*x^2+16*x^4)
   y*(7*r^2-6)*(r^4-12*r^2*x^2+16*x^4)
   -7*r^6*x+56*r^4*x^3-112*r^2*x^5+64*x^7
   -y*(r^6-24*r^4*x^2+80*r^2*x^4-64*x^6)];
%Polar representation of 36 terms of standard Zernike polynomials
   Z0=[1
   r0*cos(theta)
   r0*sin(theta)
   2*r0^2-1
   r0^2*cos(2*theta)
   r0^2*sin(2*theta)
   (3*r0^3-2*r0)*cos(theta)
   (3*r0^3-2*r0)*sin(theta)
   r0^3*cos(3*theta)
   r0^3*sin(3*theta)
   6*r0^4-6*r0^2+1
   (4*r0^4-3*r0^2)*cos(2*theta)
   (4*r0^4-3*r0^2)*sin(2*theta)
   r0^4*cos(4*theta)
   r0^4*sin(4*theta)
   (10*r0^5-12*r0^3+3*r0)*cos(theta)
   (10*r0^5-12*r0^3+3*r0)*sin(theta)
   (5*r0^5-4*r0^3)*cos(3*theta)
   (5*r0^5-4*r0^3)*sin(3*theta)
   r0^5*cos(5*theta)
   r0^5*sin(5*theta)
   20*r0^6-30*r0^4+12*r0^2-1
   (15*r0^6-20*r0^4+6*r0^2)*cos(2*theta)
   (15*r0^6-20*r0^4+6*r0^2)*sin(2*theta)
   (6*r0^6-5*r0^4)*cos(4*theta)
   (6*r0^6-5*r0^4)*sin(4*theta)
   r0^6*cos(6*theta)
   r0^6*sin(6*theta)
   (35*r0^7-60*r0^5+30*r0^3-4*r0)*cos(theta)
   (35*r0^7-60*r0^5+30*r0^3-4*r0)*sin(theta)
   (21*r0^7-30*r0^5+10*r0^3)*cos(3*theta)
   (21*r0^7-30*r0^5+10*r0^3)*sin(3*theta)
   (7*r0^7-6*r0^5)*cos(5*theta)
   (7*r0^7-6*r0^5)*sin(5*theta)
   r0^7*cos(7*theta)
   r0^7*sin(7*theta)];
%Norms of Zernike polynomials
N=[pi;pi/4;pi/4;pi/3;pi/6;pi/6;pi/8;pi/8;pi/8;pi/8;pi/5;pi/10;pi/10;pi/
10;pi/10;pi/12;pi/12;pi/12;pi/12;pi/12;pi/12;pi/7;pi/14;pi/14;pi/14;
pi/14;pi/14;pi/14;pi/16;pi/16;pi/16;pi/16;pi/16;pi/16;pi/16;pi/16];
```

(Table continued)

```
%Transformation matrix
T=sym(zeros(36,36));
matlabpool open;
parfor i=1:36
  for j=1:36
    T(i,j)=int(int(expand(Z0(i)*Z(j)*r0/N(i)),r0,0,1),theta,0,2*pi);
  end
end
matlabpool close;
```

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